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On designs related to coherent configurations of type $\binom{2\ 2}{\ \ 4}$

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Abstract

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Coherent configurations of type $\binom{2\ 2}{\ \ 4}$ correspond to certain complementary pairs of 2-designs with two or three intersection sizes, which are investigated in this paper. We establish relations on the parameters of such designs, and use them to show the Witt design $\mathcal{S}(5, 8, 24)$ is determined by the association scheme on its blocks, and to characterize the family of designs based on systems of linked symmetric designs.

0. Introduction

Coherent configurations of type $\binom{2\ 2}{\ \ n}$ (which we will write as $(2, 2; n)$) correspond to complementary pairs of 2-designs. These are symmetric designs for $n = 2$, and quasi-symmetric designs for $n = 3$. The correspondence between pairs of quasi-symmetric designs and coherent configurations was exploited in [10], which is also a basic reference for coherent configurations (c.c.'s).

In this paper we study the designs corresponding to c.c.'s of type $(2, 2; 4)$. This study was motivated by some of the examples which are listed in Section 3; in particular, the Witt design $\mathcal{S}(5, 8, 24)$ is such a design.

A c.c. of type $(2, 2; 4)$ corresponds to a pair of 2-designs each having either two or three block intersection sizes, and we will refer to the design with the smaller block size (or either one if they have the same block size) as the design associated with the c.c. If the design is quasi-symmetric, then the c.c. of type $(2, 2; 4)$ is a refinement (see [10]) of the c.c. of type $(2, 2; 3)$, and we refer to it as a $(2, 2; 4)$ refinement of the quasi-symmetric design. A design with three intersection sizes which satisfies the conditions necessary to be associated with a c.c. of type $(2, 2; 4)$ is said to be *strictly coherent*

A c.c. of type $(2,2;4)$ has a subconfiguration of type (4) whose objects are the blocks of the design; this is called the *block configuration*. C.c.'s of type (4) are the (not necessarily symmetric) association schemes with three associate classes, and have been studied in [11] and [15]. We only consider the case where the block configuration is symmetric.

Standard techniques are used in Section 4 to derive parameter equations for the coherent configuration, and hence for the related designs. These equations are then used in the remainder of the paper.

Section 5 considers the question of what we can say about a c.c. of type $(2,2;4)$ with given block configuration. In particular, we show that $\mathcal{S}(5,8,24)$ is characterized as a strictly coherent design by the intersection matrices of its block configuration, and that a rectangular lattice scheme is never the block configuration of a c.c. of type $(2,2;4)$.

In Section 6, we consider c.c.'s of type $(2,2;4)$ whose block configurations are imprimitive association schemes. We characterize the designs based on systems of linked symmetric designs (Example 5), and show that no similar family of examples arises from quasi-symmetric designs.

1. Definitions

Let X be a finite set, and $f_i \subseteq X^2$, for $i \in I$, be a binary relation on X, where I is an indexing set. $\mathcal{C} = (X, (f_i)_{i\in I})$ is a *coherent configuration* (c.c.) if the following conditions are satisfied:

(1.1) $(f_i)_{i\in I}$ is a partition of X^2.

(1.2) For all $i \in I$, $f_i^t = f_{i^*}$ for some $i^* \in I$.

(3) $f_i \cap \operatorname{diag}(X^2) \neq \emptyset$ implies $f_i \subseteq \operatorname{diag}(X^2)$.

(4) If $(x,y) \in f_k$, then $|\{z: (x,z) \in f_i \text{ and } (z,y) \in f_j\}|$ is a constant depending only on i, j, and k (for $i, j, k \in I$). This constant is denoted p_{ij}^k.

The c.c. is *homogeneous* if $\operatorname{diag}(X^2) = f_i$ for some i, *symmetric* if $f_i^t = f_i$ for all i, and *commutative* if $p_{ij}^k = p_{ji}^k$ for all $i, j, k \in I$.

There is a *standard partition* $(X_\alpha)_{\alpha\in\Omega}$ of X such that for all $i \in I$, $f_i \subseteq X_\alpha \times X_\beta$ for some $\alpha, \beta \in \Omega$ and $\operatorname{diag}(X_\alpha^2) = f_i$ for some i. Let $I^{\alpha\beta} = \{i: f_i \subseteq X_\alpha \times X_\beta\}$. $\mathcal{C}$ is said to have *rank* $|I|$ and *type* $(|I^{\alpha\beta}|)$. Note that the type is a symmetric matrix, so the entries below the diagonal are usually omitted.

The *fibers* of $\mathcal{C}$ are $\mathcal{C}^\alpha = (X_\alpha, (f_i)_{i\in I^{\alpha\alpha}})$, which are homogeneous c.c.'s with rank $|I^{\alpha\alpha}|$. If $f_i \in I^{\alpha\beta}$ and $x \in X_\alpha$, then $n_i = |\{y: (x,y) \in f_i\}|$ is a constant.

In terms of this terminology, the association schemes with n associate classes are precisely the commutative (which implies homogeneous) rank $n+1$ c.c.'s.

A c.c. $\mathcal{C}' = (X', (f_i')_{i\in I'})$ is a *refinement* of $\mathcal{C} = (X, (f_i)_{i\in I})$ if $X = X'$, and for all $j \in I$, there is a subset I_j' of I' such that $f_j = \bigcup_{i\in I_j'} f_i'$. In this case, $\mathcal{C}$ is said to be obtained from $\mathcal{C}'$ by *fusion*. If $\mathcal{C}'$ is of type $(r_{\alpha\beta})$, it is an $(r_{\alpha\beta})$ refinement of $\mathcal{C}$.

Corresponding to the relations f_i are the $|X| \times |X|$ *adjacency matrices* A_i, where $(A_i)_{x,y} = 1$ if $(x, y) \in f_i$, and 0 otherwise. The A_i are a basis for the *adjacency algebra* $\mathscr{A}(\mathscr{C})$. The regular representation of this algebra maps the matrices A_i to the *intersection matrices* M_i which are $|I| \times |I|$ matrices with $(M_i)_{s,t} = p_{si}^t$.

A basic reference for coherent configurations is [10], and the reader is referred to that paper for more details.

Theorem 1.1 [10]. *For a c.c.* $\mathscr{C}$ *as above,*

(1.5) $p_{ij}^k n_k = p_{kj^*}^i n_i$,

(1.6) *If* $j \in I^{\beta\gamma}$ *and* $k \in I^{\alpha\gamma}$, *then* $\sum_{i \in I} p_{ij}^k = n_{j^*}$,

(1.7) $M_i M_j = \sum p_{ij}^k M_k$.

We will make frequent use of this theorem without explicit reference.

The algebra $\mathscr{A}(\mathscr{C})$ has irreducible representations, and we will consider these and their multiplicities in the regular representation. In the case of a homogeneous c.c., these are just the eigenvalues of the adjacency matrices and their multiplicities. The representations satisfy orthogonality relations and Krein conditions; see [10] for more details.

The subject of this paper is c.c.'s of type $\binom{2\ \ 2}{\ \ 4}$ (which we will write as $(2, 2; 4)$). These turn out to be 2-designs satisfying certain properties. [8] is a reference for material on t-(v, k, λ) designs. We will use the term *design* to refer to a 2-design, and assume that designs have no repeated blocks. If $\mathscr{D}$ is a t-design, $\mathscr{D}_p$ and $\mathscr{D}^p$ will denote the derived and residual designs respectively.

A design is *symmetric* if any two distinct blocks have the same number of points in common; it is *quasi-symmetric* if the number of points in common with two distinct blocks takes on two different values. The possible values for the number of points in common with two distinct blocks will be referred to as the (*block*) *intersection sizes of the design.*

If a design $\mathscr{D}$ has block intersection size x_i, the *block graph* corresponding to x_i is the graph with vertices the blocks of $\mathscr{D}$ and adjacency defined by intersecting in x_i points. For a quasi-symmetric design, the two block graphs are strongly regular, where a graph is said to be *strongly regular* if it is regular and the number of vertices adjacent to two given vertices depends only on whether they are adjacent or not.

2. Type (2, 2; 4)

Let $\mathscr{C} = (X, (f_i)_{i \in I})$ be a c.c. of type $(2, 2; 4)$. We can write $X = X_1 \cup X_2$, and number the relations so the fibers are

$$\mathscr{C}^1 = (X_1, (f_4, f_5)) \quad \text{with } \operatorname{diag}(X_1^2) = f_4,$$
$$\mathscr{C}^2 = (X_2, (f_0, f_1, f_2, f_3)) \quad \text{with } \operatorname{diag}(X_2^2) = f_0,$$

and $X_1 \times X_2 = f_6 \cup f_7$, $X_2 \times X_1 = f_8 \cup f_9$, with $f_6^t = f_8$ and $f_7^t = f_9$. $\mathscr{C}^2$ is a c.c. of type (4) called the *block configuration*. It is always commutative since it has rank $\leqslant 5$ [12], and hence is an association scheme with 3 associate classes. C.c.'s of type (4) have been studied in [11] and [15]. In this paper, we will consider only c.c.'s of type (2, 2; 4) with symmetric block configuration. The graphs $\Gamma_i = (X_2, f_i)$ will be referrred to as the *block graphs*; this is not exactly the same as our use of the term earlier, but should cause no confusion.

Proposition 2.1. *(X_1, X_2, f_6) is a 2-design with at most three intersection sizes.*

Proof. Call the elements of X_1 points and the elements of X_2 blocks. Two distinct points are in p_{68}^5 blocks, a block contains p_{68}^0 points, and two blocks intersect in p_{68}^1, p_{68}^2, or p_{68}^3 points. □

Note that (X_1, X_2, f_7) is the complement of this design and hence also a 2-design with at most three intersection sizes. We number f_6 and f_7 so that (X_1, X_2, f_6) is a 2-(v, k, λ) design with $k \leqslant v/2$, and refer to this as the design associated with the c.c.

We next examine the conditions under which a design is associated with a c.c. of type (2, 2; 4).

Suppose $\mathscr{D}$ is a 2-design with three symmetric irreflexive binary relations f_1, f_2, and f_3 defined on the blocks which satisfy the following conditions:

(2.1) For any two distinct blocks B_1 and B_2, $(B_1, B_2) \in f_i$ for exactly one $i \in \{1, 2, 3\}$.

(2.2) If $(B_1, B_2) \in f_i$, then the number of points incident with both B_1 and B_2 is x_i, and $x_2 \neq x_3$.

(2.3) If p is a point and B a block, the number of blocks incident with p having relation f_1 to B depends only on whether p is incident with B or not. This number is denoted N if p is incident with B, and P if p is not incident with B.

(2.4) If B_1 and B_2 are blocks, then the number of blocks C with $(B_1, C) \in f_1$ and $(C, B_2) \in f_1$ depends only on the relation between B_1 and B_2. It is denoted p_{11}^i if $(B_1, B_2) \in f_i$, $i = 1, 2, 3$.

A design associated with a c.c. of type (2, 2; 4) clearly satisfies (2.1)–(2.4). We will show that assuming these regularity conditions is enough to go from a design to a (2, 2; 4) c.c.

The design and the relations f_1, f_2, and f_3 give rise to a configuration in a natural way. Let $X = \mathscr{P} \cup \mathscr{B}$, where $\mathscr{P}$ is the set of points of $\mathscr{D}$ and $\mathscr{B}$ is the set of blocks, and define additional relations by

$$f_0 = \operatorname{diag}(\mathscr{B}^2), \qquad f_4 = \operatorname{diag}(\mathscr{P}^2), \qquad f_5 = \mathscr{P}^2 - \operatorname{diag}(\mathscr{P}^2),$$

$$f_6 = \{(p, B)\text{: point } p \text{ is incident with block } B\}, \qquad f_7 = \mathscr{P} \times \mathscr{B} - f_6,$$

$$f_8 = f_6^t, \qquad f_9 = f_7^t.$$

Theorem 2.2. *$(X, (f_i)_{i=0}^{9})$ is a coherent configuration of type* $(2, 2; 4)$.

Proof. $(X, (f_i)_{i=0}^{9})$ clearly satisfies (1.1)–(1.3) for a c.c. It remains to check (1.4), that is, that the p_{ij}^{k}'s are well-defined; this is equivalent to showing that the linear span of the relations is a subalgebra of the algebra of $v+b \times v+b$ matrices (see [10]).

Let C be the incident matrix of the design, and A_i be the matrix of the relation f_i. As usual, J is the all 1 matrix of appropriate size, so $A_3 = J - I - A_1 - A_2$. The relations f_i correspond to matrices as follows (we assume X is arranged with the blocks before the points);

$$f_0\colon \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \quad f_1\colon \begin{pmatrix} A_1 & 0 \\ 0 & 0 \end{pmatrix} \quad f_2\colon \begin{pmatrix} A_2 & 0 \\ 0 & 0 \end{pmatrix} \quad f_3\colon \begin{pmatrix} A_3 & 0 \\ 0 & 0 \end{pmatrix}$$

$$f_4\colon \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \quad f_5\colon \begin{pmatrix} 0 & 0 \\ 0 & J-I \end{pmatrix}$$

$$f_6\colon \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \quad f_7\colon \begin{pmatrix} 0 & 0 \\ J-C & 0 \end{pmatrix} \quad f_8\colon \begin{pmatrix} 0 & C^t \\ 0 & 0 \end{pmatrix} \quad f_9\colon \begin{pmatrix} 0 & J-C^t \\ 0 & 0 \end{pmatrix}.$$

The result follows from the following matrix calculations. $(\mathcal{P}, \mathcal{B}, f_6)$ is a 2-design, so the following equations hold:

$$CC^t = (r-\lambda)I + \lambda J, \qquad CJ = rJ, \qquad JC = kJ.$$

We also have, from (2.2)–(2.4),

$$CA_1 = (N-P)C + PJ,$$

$$C^tC = (k-x_3)I + (x_1-x_3)A_1 + (x_2-x_3)A_2 + x_3J,$$

$$A_1^2 = (n_1 - p_{11}^3)I + (p_{11}^1 - p_{11}^3)A_1 + (p_{11}^2 - p_{11}^3)A_2 + p_{11}^3J.$$

From $(CC^t)C = C(C^tC)$,

$$(r-\lambda)C + \lambda kJ = (k-x_3)C + (x_1-x_3)CA_1 + (x_2-x_3)CA_2 + x_3rJ.$$

thus $CA_2 \in \langle J, C\rangle$. Similarly, computing C^tCA_1 and A_1C^tC, we get $A_1A_2 = A_2A_1 \in \langle I, J, A_1, A_2\rangle$. □

We can obviously replace f_1 by f_2 or f_3 in the conditions (2.1)–(2.4) by renumbering the relations. If the design is quasi-symmetric, we will usually assume $x_2 = x_3$, and check that the conditions hold for f_2.

Conditions (2.3) and (2.4) are both required for a design to be associated with a c.c. of type $(2, 2; 4)$ in this way. The inversive planes of odd order satisfy (2.3) but not (2.4), and the 2-$(56, 12, 3)$ design in [3] satisfies (2.4) but not (2.3).

3. Examples

In this section, we list the known examples of c.c.'s of type $(2, 2; 4)$. In each case, we describe the associated design (and the refinement, in the case of two intersection sizes).

Example 1 (*3-subsets*). The design on v points whose blocks are all subsets of order 3.

Example 2 (*3-dimensional subspaces*). Let V be a d-dimensional vector space over $\mathrm{GF}(q)$, $d \geqslant 6$. Define a design by taking the points to be the 1-dimensional subspaces of V, and the blocks to be the 3-dimensional subspaces.

Example 3 (*(d-3)-dimensional subspaces*). The design whose points are the 1-dimensional subspaces and blocks are the $(d-3)$-dimensional subspaces of V (V as above, $d \geqslant 7$).

Example 4 (*Inversive planes of even order*). A finite inversive plane of even order is a 3-$(j^2+1, j+1, 1)$ design with j a power of 2 (see, e.g. [9]). For $j \geqslant 4$, such a design has three intersection sizes—0, 1, and 2—and is associated with a c.c. of type $(2, 2; 4)$.

Example 5 (*Designs based on systems of linked symmetric designs*). Let $\{\Omega_1, \Omega_2, \ldots, \Omega_l\}$ be a system of linked symmetric designs, i.e. a collection of sets together with incidence relations F_{ij} such that $(\Omega_i, \Omega_j, F_{ij})$ is a symmetric 2-(v, k, λ) design for all $i \neq j$. (See [6], where these are called systems of linked projective designs.) We define a design based on this system as follows: Let Ω_l be the point set, $\Omega_1 \cup \Omega_2 \cup \cdots \cup \Omega_{l-1}$ be the block set, and define incidence by $p \in \Omega_l$ is incident with $B \in \Omega_i$ if $p \in B$ in $(\Omega_l, \Omega_i, F_{li})$, $1 \leqslant i \leqslant l-1$. This is based on an example of Mathon in [15].

Example 6 (*Designs based on an affine design and a quasi-symmetric design*). Let $\mathscr{D}_1$ be a 2-(v_1, k_1, λ_1) design which is affine, i.e. the blocks of $\mathscr{D}_1$ can be partitioned into parallel classes such that any two nonparallel blocks intersect in a points; say there are v_2 blocks per parallel class. Let $\mathscr{D}_2$ be a quasi-symmetric 2-(v_2, k_2, λ_2) design. We define a design $\mathscr{D}$ by taking the points to be the points of $\mathscr{D}_1$, and the blocks to be unions of k_2 parallel blocks of $\mathscr{D}_1$, chosen as follows: The blocks of a parallel class may be considered to be the points of $\mathscr{D}_2$; take the unions of all k_2-subsets which correspond to blocks of $\mathscr{D}_2$. The simplest example of this is: the points are the points of an affine plane, and the blocks are pairs of parallel lines. In general, $\mathscr{D}$ is a 2-$(v_1, k_1k_2, \lambda_1 r_2 + (r_1 - \lambda_1)\lambda_2)$ design with three intersection sizes which is associated with a c.c. Note that if we instead take $\mathscr{D}_2$

to be a symmetric design, then $\mathscr{D}$ is strongly resolvable, and this is Hughes and Piper's construction in [14].

Example 7 (*Sporadic examples*). (a) $\mathscr{S}(4, 5, 11)$,

(b) $\mathscr{S}(4, 5, 11)^p$,

(c) $\mathscr{S}(5, 8, 24)$,

(d) $\mathscr{S}(5, 8, 24)^p$,

(e) $\mathscr{D}_p$, where $\mathscr{D}$ is the design based on the codewords of weight 12 in the Golay code of length 24.

(f) Let $\mathscr{D}$ be the design based on the codewords of weight 12 in the extended quadratic residue code of length 48 over GF(2). Then $\mathscr{D}_p$ is a 4-(47, 11, 8) design (see [1]) with three intersection sizes which is associated with a c.c. of type (2, 2; 4).

Example 8 (*Pair designs with a parallelism*). The design on v points whose blocks are the pairs of points is a quasi-symmetric 2-(v, 2, 1) design. If v is even, there is a parallelism on the blocks [2], and this defines a (2, 2; 4) refinement.

Example 9 (*Lines in affine space*). Let V be a d-dimensional vector space over GF(q), with $d \geq 3$. The points and lines of V define a quasi-symmetric 2-(q^d, q, 1) design, and the refinement is given by the usual parallelism.

Example 10 (*Unitals in* PG(2, q^2) *with a quasi-parallelism*). If V is a 3-dimensional vector space over GF(q^2) and f is a hermitian inner product on V, the isotropic and non-isotropic points of PV form a 2-(q^3+1, $q+1$, 1) design called a unital with orthogonality as incidence. The design is quasi-symmetric, and orthogonality of blocks defines a (2, 2; 4) refinement on intersection size 0. The new relation on the blocks is a quasi-parallelism in the sense of [11], where these examples were first defined.

Example 11 (*Unitals in* PG(2, q^2) *with a parallelism*). Let $\mathscr{D}$ be the unital defined in Example 9. A degenerate line l of PV defines a parallelism (and hence a (2, 2; 4) refinement) whose q^2 parallel classes are $\pi_x = \{x\} \cup \{L^\perp : L$ is a nondegenerate line containing $x\}$, where x runs through the q^2 non-isotropic points of l. This is a generalization of an example of Brouwer [4].

Example 12 (*Refinements of a strongly resolvable design based on a strongly regular graph*). Let $\mathscr{D}$ be a strongly resolvable design, so $\mathscr{D}$ is quasi-symmetric and the blocks may be partitioned into d classes such that the intersection size of two blocks depends only on whether they are in the same class or not. Let Γ be a strongly regular graph on d points. Then Γ defines a (2, 2; 4) refinement as follows: Let $\mathscr{B}_1, \ldots, \mathscr{B}_d$ be the block classes. Define a new relation f_2 by $(B, C) \in f_2$ if $B \in \mathscr{B}_i$, $C \in \mathscr{B}_j$, $i \neq j$, and i is adjacent to j in Γ.

4. Parameter equations

This section develops equations which hold for the parameters of a coherent configuration of type (2,2;4) with symmetric rank 4 fiber. We will use the following notation.

Let $\mathscr{C}=(X,(f_i)_{i\in I})$ be a c.c. of type (2,2;4) over $I=\{0,1,\ldots,9\}$ with standard partition (X_1,X_2) such that $I^{11}=\{4,5\}$, $I^{22}=\{0,1,2,3\}$, $I^{12}=\{6,7\}$, and $I^{21}=\{8,9\}$ with $6^*=8$ and $7^*=9$. Thus $\mathscr{C}^2=(X_2,(f_0,f_1,f_2,f_3))$ is a rank 4 symmetric c.c. (and an association scheme with 3 associate classes). Let $n_i=p^0_{ii}$, $i=1,2,3$. The character multiplicity table of $\mathscr{C}^2$ may be written as follows:

$$\begin{array}{cccc|c} 1 & n_1 & n_2 & n_3 & 1 \\ 1 & \theta_{11} & \theta_{12} & \theta_{13} & z_1 \\ 1 & \theta_{21} & \theta_{22} & \theta_{23} & z_2 \\ 1 & \theta_{31} & \theta_{32} & \theta_{33} & z_3. \end{array}$$

(X_1,X_2,f_6) and (X_1,X_2,f_7) are a complementary pair of designs; renumber f_6 and f_7 if necessary so (X_1,X_2,f_6) is the design associated with the configuration, that is (X_1,X_2,f_6) has $k\leqslant v/2$. This design has parameters $v=|X_1|$, $b=|X_2|$, $k=p^0_{86}$, $r=p^4_{68}$, and $\lambda=p^5_{68}$. There are at most 3 block intersection sizes, denoted $x_1=p^1_{86}$, $x_2=p^2_{86}$, and $x_3=p^3_{86}$. It will be shown later that there are at least two block intersection sizes. If there are exactly two intersection sizes, the associated design is quasi-symmetric, and the configuration is a refinement of the (2,3;3) c.c. for that design.

Other notation which will be used is

$$N_1=p^6_{61},\qquad N_2=p^6_{62},\qquad N_3=p^6_{63},$$
$$P_1=p^7_{61},\qquad P_2=p^7_{62},\qquad P_3=p^7_{63}.$$

In terms of the design, these may be interpreted as follows: Given a block B and a point p, the number of blocks C containing p such that $(B,C)\in f_i$ $(i=1,2,3)$ is N_i if $p\in B$ and P_i if $p\notin B$.

We now state the theorem relating these parameters. This is not an exhaustive list of the parameter relations following from coherent configuration conditions; see also [11] for equations involving only the block configuration.

Theorem 4.1. *The parameters of a c.c. of type* (2,2;4) *with symmetric block configuration satisfy the following conditions*:

(4.1) $x_in_i=N_ik$, $i=1,2,3$,

(4.2) $(k-x_i)n_i=P_i(v-k)$, $i=1,2,3$,

(4.3) $1+N_1+N_2+N_3=r$,

(4.4) $P_1+P_2+P_3=r$,

(4.5) $k+x_1N_1+x_2N_2+x_3N_3=r+\lambda(k-1)$,

(4.6) $x_iN_i+(k-x_i)P_i=k+x_1p^i_{1i}+x_2p^i_{2i}+x_3p^i_{3i}$, $i=1,2,3$,

(4.7) $x_iN_j+(k-x_i)P_j=x_1p^i_{1j}+x_2p^i_{2j}+x_3p^i_{3j},\ i\neq j,\ i, j\in\{1, 2, 3\}$,
(4.8) $z_1=v-1$,
(4.9) $\theta_{1i}=N_i-P_i,\ i=1, 2, 3$,
(4.10) $k+x_1\theta_{11}+x_2\theta_{12}+x_3\theta_{13}=r-\lambda$,
(4.11) $k+x_1\theta_{21}+x_2\theta_{22}+x_3\theta_{23}=0$,
(4.12) $k+x_1\theta_{31}+x_2\theta_{32}+x_3\theta_{33}=0$.

These equations can be used to show that the associated design has at least two intersection sizes.

Proposition 4.2. *There is no c.c. of type* (2, 2; 4) *with* $x_1=x_2=x_3$.

Proof. Suppose $x_1=x_2=x_3$. Then by (4.6) and (4.7) we have

$$x_1N_1+(k-x_1)P_1=k+x_1(p^1_{11}+p^1_{21}+p^1_{31})=k+x_1(n_1-1)$$

and

$$x_1N_1+(k-x_1)P_1=x_1(p^2_{11}+p^2_{21}+p^2_{31})=x_1n_1.$$

Therefore $k=x_1$ and every block must have the same point set, so every point is incident with every block. But then $f_7=\emptyset$, a contradiction. □

Proof of Theorem 4.1. The intersection matrices for $\mathscr{C}$ will be given in terms of the matrices $M^x_j=(p^t_{sj})_{s\in I^{x\alpha}, t\in I^{x\beta}}$. The matrix M^x_i is the $(I^{x\alpha}, I^{x\beta})$ block of M_i if $i\in I^{\alpha\beta}$, where M_i is blocked according to the standard partition. Using the parameters defined earlier, these matrices are

$M^1_0=I,\qquad M^2_0=I,$

$$M^1_1=\begin{pmatrix} N_1 & P_1\\ n_1-N_1 & n_1-P_1\end{pmatrix},\qquad M^2_1=\begin{pmatrix} 0 & 1 & 0 & 0\\ n_1 & p^1_{11} & p^2_{11} & p^3_{11}\\ 0 & p^1_{21} & p^2_{21} & p^3_{21}\\ 0 & p^1_{31} & p^2_{31} & p^3_{31}\end{pmatrix},$$

$$M^1_2=\begin{pmatrix} N_2 & P_2\\ n_2-N_2 & n_2-P_2\end{pmatrix},\qquad M^2_2=\begin{pmatrix} 0 & 0 & 1 & 0\\ 0 & p^1_{12} & p^2_{12} & p^3_{12}\\ n_2 & p^1_{22} & p^2_{22} & p^3_{22}\\ 0 & p^1_{32} & p^2_{32} & p^3_{32}\end{pmatrix},$$

$$M^1_3=\begin{pmatrix} N_3 & P_3\\ n_3-N_3 & n_3-P_3\end{pmatrix},\qquad M^2_3=\begin{pmatrix} 0 & 0 & 0 & 1\\ 0 & p^1_{13} & p^2_{13} & p^3_{13}\\ 0 & p^1_{23} & p^2_{23} & p^3_{23}\\ n_3 & p^1_{33} & p^2_{33} & p^3_{33}\end{pmatrix},$$

$M^1_4=I,\qquad M^2_4=I,$

$$M_5^1=\begin{pmatrix}0 & 1\\ v-1 & v-2\end{pmatrix},\qquad M_5^2=\begin{pmatrix}k-1 & k\\ v-k & v-k-1\end{pmatrix},$$

$$M_6^1=\begin{pmatrix}1 & 0\\ k-1 & k\end{pmatrix},\qquad M_6^2=\begin{pmatrix}k & x_1 & x_2 & x_3\\ 0 & k-x_1 & k-x_2 & k-x_3\end{pmatrix},$$

$$M_7^1=\begin{pmatrix}0 & 1\\ v-k & v-k-1\end{pmatrix},\quad M_7^2=\begin{pmatrix}0 & k-x_1 & k-x_2 & k-x_3\\ v-k & v-2k+x_1 & v-2k+x_2 & v-2k+x_3\end{pmatrix},$$

$$M_8^1=\begin{pmatrix}r & \lambda\\ 0 & r-\lambda\end{pmatrix},\qquad M_8^2=\begin{pmatrix}1 & 0\\ N_1 & P_1\\ N_2 & P_2\\ N_3 & P_3\end{pmatrix},$$

$$M_9^1=\begin{pmatrix}0 & r-\lambda\\ b-r & b-2r+\lambda\end{pmatrix},\qquad M_9^2=\begin{pmatrix}0 & 1\\ n_1-N_1 & n_1-P_1\\ n_2-N_2 & n_2-P_2\\ n_3-N_3 & n_3-P_3\end{pmatrix}.$$

Note that since we assume the block configuration is symmetric, $p_{ij}^k=p_{ji}^k$, for $0\leqslant i,j,k\leqslant 3$.

Equation (4.1)–(4.7) follow from these matrices and Theorem 1.1.

The character-multiplicity table for $\mathscr{C}^2$ was give earlier. The character multiplicity table for $\mathscr{C}^1$ is

$$\begin{array}{cc|c} 1 & v-1 & 1\\ 1 & -1 & v-1. \end{array}$$

The only possibility is that $\mathscr{C}$ has two representations of degree 1 and two of degree 2. Thus one of the z_i must be $v-1$; renumber if necessary so this is z_1. Then the degrees for $\mathscr{C}$ are

$$e_0=e_{10}+e_{20}=2,\qquad e_1=e_{11}+e_{21}=2,$$
$$e_3=e_{23}=1,\quad\text{and}\quad e_4=e_{24}=1.$$

The multiplicities are 1, $z_1=v-1$, z_2, and z_3. The irreducible representations Δ_0, Δ_1, Δ_2, and Δ_3 are given in Table 1, and equations (4.8)–(4.12) follow from this. The E_{ij} are the 2×2 matrix units, and $\alpha_1=(kr)^{\frac{1}{2}}$, $\alpha_2=(n_1-N_1)(kr)^{\frac{1}{2}}/P_1$, and $\beta_1=-\beta_2=(r-\lambda)^{\frac{1}{2}}$. $\square$

A coherent configuration of type (2, 2; 4) is associated with a design with either two or three intersection sizes. For each case, we can use theorem 4.1 to express all the parameters in terms of a small number of them.

A *strictly coherent design* of type $(2,2;c+1)$ is a 2-design $(\mathscr{P},\mathscr{B},F)$ with c intersection sizes $x_1,x_2,\ldots,x_c$ such that $X=\mathscr{P}\cup\mathscr{B}$ together with the natural relations is a c.c. of type $(2,2;c+1)$. That is, let $f_0=\mathrm{diag}(\mathscr{B}^2)$, $f_i=\{(B,c)\colon |B\cap$

Table 1
Irreducible representations

	Δ_0	Δ_1	Δ_2	Δ_3
A_0	E_{11}	E_{11}	1	1
A_1	n_1E_{11}	$\theta_{11}E_{11}$	θ_{21}	θ_{31}
A_2	n_2E_{11}	$\theta_{12}E_{11}$	θ_{22}	θ_{32}
A_3	n_3E_{11}	$\theta_{13}E_{11}$	θ_{23}	θ_{33}
A_4	E_{22}	E_{22}	0	0
A_5	$(v-1)E_{22}$	$-E_{22}$	0	0
A_6	α_1E_{21}	β_1E_{21}	0	0
A_7	α_2E_{21}	β_2E_{21}	0	0
A_8	α_1E_{12}	β_2E_{12}	0	0
A_9	α_2E_{12}	β_2E_{12}	0	0
mult.	1	z_1	z_2	z_3

$C|=x_i\}$, $f_{c+1}=\operatorname{diag}(\mathscr{P}^2)$, $f_{c+2}=\mathscr{P}^2-f_{c+1}$, $f_{c+3}=F$, $f_{c+4}=\mathscr{P}\times\mathscr{B}-F$, $f_{c+5}=f^t_{c+3}$, $f_{c+6}=f^t_{c+4}$; then $(X,(f_i)_{i=0}^{c+6})$ is a c.c. of type $(2,2;c+1)$. Examples 1–7 are strictly coherent designs of type $(2,2;4)$.

For strictly coherent designs of type $(2,2;4)$, we can express all of the parameters in terms of seven of them.

Proposition 4.3. *The parameters of a strictly coherent design of type* $(2,2;4)$ *may be expressed in terms of* b, v, k, x_1, x_2, x_3, *and* p^1_{31} *as follows*:

$$r=\frac{bk}{v},\qquad \lambda=\frac{bk(k-1)}{v(v-1)},$$

$$n_1=\frac{1}{(x_1-x_2)(x_1-x_3)}(kr-k^2+\lambda k(k-1)+x_2x_3(b-1)-(x_2+x_3)k(r-1)),$$

$$n_2=\frac{1}{x_2-x_3}(k(r-1)-(x_1-x_3)n_1-x_3(b-1)),\qquad n_3=b-(1+n_1+n_2),$$

$$N_i=x_in_i/k,\qquad P_i=\frac{(k-x_i)n_i}{(v-k)},\qquad i=1,2,3,$$

$$p^1_{21}=\frac{1}{x_1-x_2}(k+x_1(n_1-N_1-1)-(x_1-x_3)p^1_{31}-(k-x_1)P_1),$$

$$p^3_{21}=\frac{1}{x_2-x_3}(P_1(k-x_3)-x_3(n_1-N_1)-(x_1-x_3)p^1_{31}n_1/n_3),$$

$$p^3_{22}=\frac{1}{x_2-x_3}(P_2(k-x_3)-x_3(n_2-N_2)-(x_1-x_3)p^3_{21}),$$

$$\theta_{1i}=N_i-P_i,$$

$$\theta_{21}=\frac{x+y^{\frac{1}{2}}}{2(x_2-x_3)}, \textit{ where}$$

$$x=p_{11}^1(x_2-x_3)-p_{11}^2(x_1-x_3)+p_{11}^3(x_1-x_2)),$$

$$y=x^2+4(x_2-x_3)(n_1(x_2-x_3)-p_{11}^2(k-x_3)+p_{11}^3(k-x_2)),$$

$$\theta_{31}=\frac{x-y^{\frac{1}{2}}}{2(x_2-x_3)},$$

$$\theta_{i2}=-((x_1-x_3)\theta_{i1}+k-x_3)/(x_2-x_3),\quad i=2,3,$$

$$\theta_{i3}=-1-\theta_{i1}-\theta_{i2},\quad i=2,3,$$

$$z_1=v-1,$$

$$z_2=\frac{1}{2}\left(b-v+\frac{(v-b)x-2(x_2-x_3)((v-1)(N_1-P_1)+n_1)}{y^{\frac{1}{2}}}\right),$$

$$z_3=b-v-z_2.$$

Proof. These are a direct consequence of the equations in Theorem 4.1. We mention some of the details.

n_1, n_2: Solve the system of equations

$$n_1+n_2+n_3=b-1,$$

$$x_1n_1+x_2n_2+x_3n_3=kr-k,$$

$$x_1^2n_1+x_2^2n_2+x_3^2n_3=kr+\lambda k(k-1)-k^2.$$

p_{21}^1: Use the equations

$$x_1p_{11}^1+x_2p_{21}^1=x_1N_1+(k-x_1)P_1-k-x_3p_{31}^1,$$

$$p_{11}^1+p_{21}^1=n_1-p_{31}^1-1.$$

p_{21}^3 and p_{22}^3 are obtained similarly from (4.6) and (4.7).

From the representation table, it follows that, for $i=2,3$

$$\theta_{i1}^2=n_1+p_{11}^1\theta_{i1}+p_{11}^2\theta_{i2}+p_{11}^3\theta_{i3}$$

Using this, together with the equations

$$k+x_1\theta_{i1}+x_2\theta_{i2}+x_3\theta_{i3}=0 \quad\text{and}\quad 1+\theta_{i1}+\theta_{i2}+\theta_{i3}=0,$$

we can express θ_{i2} and θ_{i3} in terms of θ_{i1}, and derive a quadratic equation with solutions θ_{21} and θ_{31}. θ_{21} is arbitrarily chosen to be the larger solution.

The formula for z_2 is obtained by solving the equations

$$z_2+z_3=b-1-z_1,$$

$$\theta_{21}z_2+\theta_{31}z_3=-n_1-\theta_{11}z_1,\quad (\text{from trace } M_1=\text{trace } A_1=0)$$

for z_2 and substituting the values for θ_{11}, θ_{21}, θ_{31}, and z_1. Note that this requires $\theta_{21}\neq\theta_{31}$. □

Proposition 4.4. *In a strictly coherent design of type* $(2, 2; 4)$, $\theta_{21} \neq \theta_{31}$.

Proof. Suppose $\theta_{21} = \theta_{31}$. Then the graph Γ_1 has three eigenvalues n_1, θ_{11}, and θ_{21}, n_1 has multiplicity 1 and eigenvector the all-1 vector, therefore Γ_1 is strongly regular and $p_{11}^2 = p_{11}^3$. Now $\theta_{21} = \theta_{31}$, which happens if and only if $y^2 = 0$, i.e.

$$0 = (p_{11}^1(x_2 - x_3) - p_{11}^2(x_1 - x_3) + p_{11}^3(x_1 - x_2))^2$$
$$+ 4(x_2 - x_3)(n_1(x_2 - x_3) - p_{11}^2(k - x_3) + p_{11}^3(k - x_2))$$

which implies $(p_{11}^1 - p_{11}^2)^2 + 4(n_1 - p_{11}^2) = 0$. Therefore $p_{11}^1 = p_{11}^2 = p_{11}^3 = n_1$, but $p_{11}^1 \leqslant n_1 - 1$, a contradiction, and $\theta_{21} \neq \theta_{31}$. □

Corollary 4.5. *For a strictly coherent design of type* $(2, 2; 4)$, *either*

(1) $x^2 + 4(x_2 - x_3)(n_1(x_2 - x_3) - p_{11}^2(k - x_3) + p_{11}^3(k - x_2))$ *is a square, and all the eigenvalues are rational, where* $x = p_{11}^1(x_2 - x_3) - p_{11}^2(x_1 - x_3) + p_{11}^3(x_1 - x_2)$, *or* (2) $(v - b)x - 2(x_2 - x_3)((v - 1)(N_1 - P_1) + n_1) = 0$, *and* $z_2 = z_3 = (b - v)/2$.

Proposition 4.6. *If a strictly coherent design of type* $(2, 2; 4)$ *has* $k = 3$, *then the design is trivial, that is, every* 3-*subset is a block.*

Proof. We have $k = 3$, $x_1 = 2$, $x_2 = 1$, and $x_3 = 0$. Note there can be no block which intersects both of two disjoint blocks in two points, so $p_{11}^3 = 0$. Now, $P_1 = n_1/(v - 3)$ and $n_1 \leqslant \binom{3}{2}\binom{v-3}{1} = 3(v - 3)$, so n_1 can only be $v - 3$, $2(v - 3)$, or $3(v - 3)$. Using Proposition 4.3, $n_1 = 3(\lambda - 1)$, so $\lambda = n_1/3 + 1$, $r = \lambda(v - 1)/(k - 1)$, and $n_2 = 3r - 6\lambda + 3$.

If $n_1 = v - 3$, then $\lambda = v/3$, $r = v(v - 1)/6$, $N_1 = 2(v - 3)/3$, $P_1 = 1$, and $n_2 = (v - 2)(v - 3)/2$. It follows that $p_{21}^1 = 2v/3$, so $p_{11}^2 = 4v/(3(v - 2))$. But this is only an integer for $v = 3$ or 6, and for each of these one of the n_i is 0, so $n_1 \neq v - 3$.

If $n_1 = 2(v - 3)$, $p_{11}^2 = 2(4v - 15)/(3(2v - 7))$, which is never an integer. Therefore, $n_1 = 3(v - 3)$. Then $b = v(v - 1)(v - 2)/6$, and every three points form a block. □

We now consider the c.c's of type $(2, 2; 4)$ associated with quasi-symmetric designs. The c.c is then a refinement of the $(2, 2; 3)$ c.c. generated by the design. We will assume in general that the relations are numbered so that $x_2 = x_3$, and say that the refinement is on intersection size x_2.

Proposition 4.7. *If* $\mathscr{C}$ *is a c.c. of type* $(2, 2; 4)$ *with* $x_2 = x_3$, *then* (X_1, X_2, f_6) *is a quasi-symmetric design with parameters* v, b, k, r, λ, *and intersection sizes* x_1 *and* x_2.

Proposition 4.8. *For a c.c. of type* $(2, 2; 4)$ *with* $x_2 = x_3$, *the parameters may be expressed in terms of* v, k, x_1, x_2, n_2, *and* p^3_{22} *as follows.*

$$\lambda = \frac{k(k-1)(k-x_1)(k-x_2)}{k^4 - 2k^3 - ((x_1 + x_2 - 1)(v-1) - 1)k^2 + x_1 x_2 v(v-1)},$$

$$b = \frac{\lambda v(v-1)}{k(k-1)}, \qquad r = \frac{bk}{v},$$

$$n_1 = \frac{1}{x_1 - x_2}(k(r-1) - x_2(b-1)), \qquad n_3 = b - (1 + n_1 + n_2),$$

$$N_i = x_i n_i / k, \quad i = 1, 2, 3, \qquad P_i = (k - x_i) n_i / (v - k),$$

$$p^3_{11} = \frac{1}{k(x_1 - x_2)} P_1(k^2 - v x_2), \qquad p^3_{21} = \frac{1}{k(x_1 - x_2)} P_2(k^2 - v x_2),$$

$$\theta_{11} = N_1 - P_1, \qquad \theta_{12} = N_2 - P_2, \qquad \theta_{13} = N_3 - P_3,$$

$$\theta_{21} = \theta_{31} = (k - x_2)/(x_2 - x_1),$$

$$\theta_{22} = \tfrac{1}{2}(p^2_{22} - p^3_{22} + y^{\frac{1}{2}}), \quad \text{where}$$

$$y = (p^2_{22} - p^3_{22})^2 + 4\left(n_2 - \frac{(k - x_2)p^1_{22}}{(x_1 - x_2)} + \frac{(k - x_1)p^3_{22}}{(x_1 - x_2)}\right),$$

$$\theta_{32} = \tfrac{1}{2}(p^2_{22} - p^3_{22} - y^{\frac{1}{2}}),$$

$$\theta_{i3} = (k - x_1)/(x_1 - x_2) - \theta_{i2}, \quad i = 2, 3,$$

$$z_1 = v - 1,$$

$$z_2 = \tfrac{1}{2}[b - v + ((v - b)(p^2_{22} - p^3_{22}) - 2((v-1)(N_2 - P_2) + n_2))/y^{\frac{1}{2}}],$$

$$z_3 = b - v - z_2.$$

Proof. These follow from Theorem 4.2 and the parameter equations for quasi-symmetric designs in [10, Section 9]. We give details for some of them.

p^3_{11}: Solve the following equation, using $x_2 = x_3$.

$$x_3 N_1 + (k - x_3)P_1 = x_1 p^3_{11} + x_2 p^3_{21} + x_3 p^3_{31},$$

$$n_1 = p^3_{11} + p^3_{21} + p^3_{21}.$$

p^3_{21}: Solve for $p^3_{12} = p^3_{21}$,

$$x_3 N_2 + (k - x_3)P_2 = x_1 p^3_{12} + x_2 p^3_{22} + x_3 p^3_{32},$$

$$n_2 = p^3_{12} + p^3_{22} + p^3_{32}.$$

The formulas for θ_{21} and θ_{31} are obtained by solving the system

$$k + x_1\theta_{i1} + x_2\theta_{i2} + x_3\theta_{i3} = 0,$$

$$1 + \theta_{i1} + \theta_{i2} + \theta_{i3} = 0, \quad i = 2, 3.$$

These also give equations for θ_{i3} in terms of θ_{i2}. From the rank 4 characters, we have

$$(\theta_{i2})^2 = n_2 + p_{22}^1\theta_{i1} + p_{22}^2\theta_{i2} + p_{22}^3\theta_{i3}, \quad i = 2, 3.$$

Substituting θ_{i1} and θ_{i3}, we get a quadratic equation with solutions θ_{22} and θ_{32}. θ_{22} is arbitrarily chosen to be the larger solution.

Solving the equations

$$z_2 + z_3 = b - 1 - z_1 \quad \text{and} \quad \theta_{22}z_2 + \theta_{32}z_3 = -n_2 - \theta_{12}z_1$$

for z_2 and substituting θ_{12}, θ_{22}, θ_{32} and z_1 gives the value for z_2. □

Note that if we renumber the relations so $x_2 \neq x_3$, the formulas for the characters and multiplicities are exactly those of Proposition 4.3.

The formula for z_2 requires that $\theta_{22} \neq \theta_{32}$. The proof is similar to that of Proposition 4.4.

Corollary 4.9. *For a* (2, 2; 4) *refinement of a quasi-symmetric design, either*

(1) $(p_{22}^2 - p_{22}^3)^2 + 4(N_2 - (k - x_2)p_{22}^1/(x_1 - x_2) + (k + x_1)p_{22}^3/(x_1 - x_2))$ *is a square and all the eigenvalues are rational, or*

(2) $(v - b)(p_{22}^2 - p_{22}^3) - 2((v - 1)(N_2 - P_2) + n_2) = 0$, *and* $z_2 = z_3 = (n - v)/2$.

5. Block configurations

The block configuration of a coherent configuration of type (2, 2; 4) is a rank 4 c.c., and hence an association scheme with three associate classes. In this section we will consider the problem of starting with a rank 4 c.c. and determining the (2, 2; 4) c.c.'s (if any) for which it is the block configuration.

The main tool is the equation $\theta_{1i} = N_i - P_i$. From this, and equations (2.5) and (2.6), it follows that for $i = 1, 2, 3$,

(5.1) $x_i = (1/v_i)(\theta_{1i}k + (n_i - \theta_{1i})k^2/v)$, and

(5.2) $N_i = \theta_{1i} + (n_i - \theta_{1i})k/v$.

Theorem 5.1. *Suppose $\mathcal{D}$ is a strictly coherent design of type* (2, 2; 4) *whose block configuration has the same intersection numbers as that of $\mathcal{S}(5, 8, 24)$. Then $\mathcal{D}$ is isomorphic to $\mathcal{S}(5, 8, 24)$ or its complement.*

Proof. A design and its complement have the same block configuration, so we assume $k \leq v/2$, and find the parameters of the design.

The intersection numbers of the block configuration determine the character-multiplicity table, which is as follows:

$$\begin{array}{rrrr|r} 1 & 280 & 448 & 30 & 1 \\ 1 & 70 & -56 & 15 & 23 \\ 1 & 4 & -12 & 7 & 252 \\ 1 & -6 & 8 & -3 & 483. \end{array}$$

One of the multiplicities must be $v-1$, and the corresponding characters are then θ_{1i}. There are three cases.

Case 1: $v=484$.

Then $\theta_{13}=3$, so $N_3=-3+3k/44$, and 44 must divide k, say $k=44a$. $k\leqslant v/2=242$; and $x_3=44a(a-1)/10$, so a must be 1 or 5. However, for each of these values of a, x_i is not an integer.

Case 2: $v=253$.

$\theta_{11}=4$, so by (5.2) $N_1=4+12k/11$. Thus $k=11a$, where $1\leqslant a\leqslant 11$. But $x_1=11a(3a+1)/70$, which is not an integer for any such a.

Case 3: $v=24$.

By the same methods, $N_1=70+35k/4$ and $k=4a$, with $a=1$ or 2. Then $x_1=a+a^2/2$, thus $a=2$, and $k=8$. The rest of the parameters can be calculated from these, and it follows from the parameters and Theorem 5.2 below that this is a 5-(24, 8, 1) design, and hence is the unique Steiner system $\mathscr{S}(5,8,24)$. $\square$

Theorem 5.2 [13]. *Let $\mathscr{D}$ be a s.c. design of type* (2, 2; 4). *For* $2\leqslant t\leqslant k$,

$$\sum_{i=0}^{3}\binom{x_i}{t}n_i\geqslant b\binom{k}{t}^2\binom{v}{t}^{-1}$$

with equality if and only if $\mathscr{D}$ is a t-design.

Remark. It follows from a theorem of Brouwer [5] that the block configuration of $\mathscr{S}(5,8,24)$ is determined as a homogeneous rank 4 c.c. by its intersection numbers.

The rectangular lattice scheme $R(n_2,n_1)$ is defined as follows (see, e.g., [15]): X is the set of pairs (i,j) with $1\leqslant i\leqslant n_1+1$, $1\leqslant j\leqslant n_2+1$, and the non-identity relations are agreeing in the first coordinate, agreeing in the second coordinate, and not agreeing in either coordinate.

Proposition 5.3. *$R(n_2,n_1)$ is not the block configuration for any c.c. of type* (2, 2; 4).

Proof. $R(n_2, n_1)$ has character multiplicity table

$$\begin{array}{cccc|c} 1 & n_1 & n_2 & n_1n_2 & 1 \\ 1 & -1 & n_2 & -n_2 & n_1 \\ 1 & n_1 & -1 & -n_1 & n_2 \\ 1 & -1 & -1 & 1 & n_1n_2 \end{array}$$

so the possibilities for v are n_1+1, n_2+1, and n_1n_2+1.

Let $\mathscr{D}$ be the design associated with a c.c. of type $(2,2;4)$ with this block configuration. Note that the block graph Γ_1 is a disjoint union of n_2+1 complete graphs on n_1+1 vertices, and this partitions the blocks into classes $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_{n_2+1}$. If p is a point of $\mathscr{D}$, and $B \in \mathscr{B}_i$, there are N_1+1 blocks of $\mathscr{B}_i$ containing p if $p \in B$ and P_1 if $p \notin B$, so $N_1+1=P_1$, and $\theta_{11}=N_1-P_1=-1$. Similarly Γ_2 is a disjoint union of complete graphs, and this implies $\theta_{12}=-1$. Therefore z_1 must be n_1n_2, and $v=n_1n_2+1$.

Now $\theta_{13}=1$, so $N_3=1+k-2k/(n_1n_2+1)$, so we can write $2k=(n_1n_2+1)a$ for some integer a. Then

$$P_1=1+N_1=(n_1+1)a/2 \leqslant n_1,$$

so $a=1$, and $k=(n_1n_2+1)/2$.

$$x_1=\frac{1}{n_1}(-k+(n_1+1)k^2/(n_1n_2+1))=k(n_1-1)/(2n_1),$$

therefore n_1 divides k. But $2k=n_1n_2+1$, so n_1 divides n_1n_2+1, and n_1 must be 1.

Thus $v=n_2+1$, and $k=(n_2+1)/2$. By (2.17),

$$x_2=(n_2+1)(n_2-1)/(4n_2),$$

which implies $n_2=1$ and $k=1$, a contradiction since $\mathscr{D}$ is a 2-design. $\square$

6. Imprimitivity

A homogeneous c.c. is *primitive* if the graphs of the relations are all connected, and *imprimitive* otherwise. We will call a $(2,2;4)$ c.c. and its associated design primitive or imprimitive as its block configuration is primitive or imprimitive.

Theorem 6.1 [7]. *The following are equivalent for a homogeneous c.c. $\mathscr{C}$ of rank s.*

(1) *$\mathscr{C}$ is imprimitive.*

(2) *A proper union of the relations f_i is an equivalence relation.*

(3) *For some integer s' with $0<s'<s-1$, $p_{ij}^k=0$ whenever $i,j \leqslant s'$, $k>s'$ (ordering the relations suitably).*

If a $(2,2;4)$ c.c. is imprimitive, we can partition the blocks into classes called *imprimitivity classes* according to the connected components of the disconnected graph; these are also the equivalence classes of the equivalence relation in (2).

If the design associated with a $(2,2;4)$ c.c. has repeated blocks, it is imprimitive since the block graph for intersection size k is a union of complete graphs. We will as usual assume no repeated blocks.

If Γ_1 is not connected, then one of the following must occur:

(6.1) $p_{11}^2 = p_{11}^3 = 0$,

(6.2) $p_{11}^2 = p_{21}^3 = 0$,

(6.3) $p_{11}^3 = p_{21}^3 = 0$.

Equations (6.2) and (6.3) are equivalent by renumbering f_2 and f_3. Analogous conditions hold if Γ_2 or Γ_3 is not connected.

We can use this to divide the imprimitive c.c.'s of type $(2,2;4)$ into three cases. It is easy to express these in terms of parameter conditions, but it is also the natural division based on combinatorial considerations.

In the quasi-symmetric case, it follows immediately from Proposition 4.8 that $p_{21}^3 = 0$ if and only if $p_{11}^2 = p_{11}^3 = 0$. The associated quasi-symmetric design is imprimitive, and we describe all refinements of such design below.

If $p_{21}^3 \neq 0$, assume w.l.o.g. that Γ_1 is not connected, so $p_{11}^2 = p_{11}^3 = 0$. We will show that this implies $x_2 \neq x_3$; if the design is quasi-symmetric, we can assume $x_1 = x_3$. The designs based on systems of linked symmetric designs are of this type, and we characterize them.

The remaining case is $p_{21}^3 = p_{11}^2 = 0$, three intersection sizes (renumbering f_2 and f_3 if necessary). It follows from the parameter conditions (Theorem 4.1) that $p_{21}^3 = p_{11}^2 = p_{11}^3 = 0$ if and only if $x_2 = x_3$, so in this case, $p_{11}^3 \neq 0$.

We will require the following definitions.

A *resolution* of a design $(\mathscr{P}, \mathscr{B}, F)$ is a partition $\mathscr{B} = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_d$ of the blocks into classes such that p is in σ_i blocks of $\mathscr{B}_i$ for all points p. It is a *σ-resolution* if $\sigma_i = \sigma$ for all i. The resolution is *inner* if any two blocks from the same class meet in ρ points; it is *outer* if any two blocks from different classes meet in β points. A *strong resolution* is a σ-resolution which is both inner and outer, and then we say the design is *strongly resolvable* (see [14] for example). A strongly resolvable design with $\rho = 0$ is also called an *affine* design. The imprimitive quasi-symmetric designs are just the strongly resolvable designs.

Case 1: $p_{21}^3 = p_{11}^2 = p_{11}^3 = 0$.

If this parameter condition holds, then $x_2 = x_3$ and the associated design is quasi-symmetric and imprimitive, hence is strongly resolvable. The c.c.'s of Example 12 satisfy this condition, and these are in fact all of them.

Theorem 6.2. *Suppose $\mathscr{C}$ is a c.c. of type $(2,2;4)$ with $p_{21}^3 = p_{11}^2 = p_{11}^3 = 0$. Then the design associated with $\mathscr{C}$ is strongly resolvable, with a refinement based on a strongly regular graph as in Example* 12.

Proof. Suppose $X_2 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_d$ is the partition of the blocks into imprimitivity classes. Then two blocks B and C intersect in x_1 points if and only if they are in the same class. From the parameter conditions, $p^1_{22} = n_2$ and $p^1_{33} = n_3$; and therefore the relationship between two blocks in different classes depends only on those classes.

Define a graph Γ on d vertices by making i adjacent to j if $(B, C) \in f_2$ for some (and hence all) $B \in \mathcal{B}_i$ and $C \in \mathcal{B}_j$. Then i is adjacent to $n_2/|\mathcal{B}_i| = n_2/(n_1+1)$ vertices, so Γ is regular. Vertices i and j are both adjacent to $p^2_{22}/(n_1+1)$ vertices if i is adjacent to j and to $p^3_{22}(n_1+1)$ vertices if not. Therefore, Γ is a strongly regular graph with the same relation to the refinement as in Example 12. □

The blocks of a strongly resolvable design in the same imprimitivity class intersect in the larger intersection size [14], hence a refinement as above is on the larger intersection size. If there is a refinement on the smaller intersection size, then $p^2_{11} = p^3_{11} = n_1$, as relation f_1 is pairs of blocks in different block classes of the strong resolution. Then $p^3_{21} + p^3_{31} = 0$ and this implies $p^2_{11} = p^3_{11} = 0$, a contradiction. Hence the refinement of the previous theorem is the only refinement of a strong resolvable design.

Case 2: $p^2_{11} = p^3_{11} = 0$, $p^3_{21} \neq 0$.

If a (2, 2; 4) c.c. satisfies $p^2_{11} = p^3_{11} = 0$, $p^3_{21} \neq 0$, then $x_2 \neq x_3$. We treat the cases of two and three intersection sizes together, and assume contrary to our practice earlier that if there are two intersection sizes, then $x_1 = x_3$.

Examples with three intersection sizes are the designs based on systems of linked symmetric designs (Example 5). For two intersection sizes, the examples with parallelisms (Examples 8, 9, and 11) are of this type; in fact any parallelism of a quasi-symmetric design such that not all blocks are parallel defines an imprimitive (2, 2; 4) refinement.

Let $X_2 = \mathcal{B}_1 \cup \mathcal{B}_2 \cup \cdots \cup \mathcal{B}_d$ be the partition of the blocks into imprimitivity classes. Using the intersection matrices, we get that $p^1_{11} = n_1 - 1$, and hence that any two blocks from the same imprimitivity class intersect in x_1 points. The following theorem characterizes Example 5.

Theorem 6.3. *If $\mathcal{D}$ is a design associated with a c.c. of type* (2, 2; 4) *such that* $p^2_{11} = p^3_{11} = 0$, $p^3_{21} \neq 0$, *and* $v = |\mathcal{B}_i|$, *then $\mathcal{D}$ is based on a system of linked symmetric designs.*

Before proving the theorem, we will describe the intersection matrices and character-multiplicity tables for any of these designs. We first show $p^2_{21} = p^3_{21} - 1$, and $P_1 = N_1 + 1$; the rest of the rank 4 intersection matrices follow from these and the parameter equations in Theorems 1.1 and 2.2.

If $B \in \mathcal{B}_i$ and $C \in \mathcal{B}_j$, the number of other blocks of $\mathcal{B}_j$ intersecting B in x_2 points is p^2_{21} if $|B \cap C| = x_2$, and p^3_{21} if $|B \cap C| = x_3$. Obviously both kinds occur since $p_{21}^3 \neq 0$, so $p_{21}^2 + 1 = p_{21}^3$. A similar argument, taking a point p and a block B of $\mathcal{B}_i$, and counting the number of blocks of $\mathcal{B}_i$ containing p gives $N_1 + 1 = P_1$.

Now $n_2 = (d-1)p_{21}^3$, $n_3 = (d-1)(n_1 + 1 - p_{21}^3)$, $(k^2 - x_1 v)n_1 = k(v-k)$, and

$$(x_2 - x_3)p_{21}^3 = (k - x_1)(k^2 - x_3 v)/(k^2 - x_1 v).$$

Proposition 6.4. *A c.c. of type* $(2,2;4)$ *with* $p_{11}^2 = p_{11}^3 = 0$, $p_{21}^3 \neq 0$, *has the following rank 4 intersection matrices, where d is the number of block classes, and* $x = n_1 + 1 - p_{21}^3$:

$$\begin{matrix} 0 & 1 & 0 & 0 \\ n_1 & n_1 - 1 & 0 & 0 \\ 0 & 0 & p_{21}^3 - 1 & p_{21}^3 \\ 0 & 0 & x & n_1 - p_{21}^3 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & 0 & p_{21}^3 - 1 & p_{21}^3 \\ p_{21}^3(d-1) & \dfrac{p_{21}^3(p_{21}^3 - 1)(d-1)}{n_1} & p_{21}^3(d-2) - \dfrac{xp_{22}^3}{p_{21}^3} & p_{22}^3 \\ 0 & \dfrac{xp_{21}^3(d-1)}{n_1} & \dfrac{xp_{22}^3}{p_{21}^3} & p_{21}^3(d-1) - p_{22}^3 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & n_1 + 1 - p_{21}^3 & n_1 - p_{21}^3 \\ 0 & \dfrac{xp_{21}^3(d-1)}{n_1} & \dfrac{xp_{22}^3}{p_{21}^3} & p_{21}^3(d-1) - p_{22}^3 \\ x(d-1) & \dfrac{x(d-1)(n_1 - p_{21}^3)}{n_1} & \dfrac{x(p_{21}^3(d-2) - p_{22}^3)}{p_{21}^3} & (x - p_{21}^3)(d-2) + p_{22}^3. \end{matrix}$$

The block configuration has character-multiplicity table

$$\begin{array}{cccc|c} 1 & n_1 & p_{21}^3(d-1) & x(d-1) & 1 \\ 1 & -1 & N_2 - P_2 & N_3 - P_3 & v-1 \\ 1 & n_1 & \dfrac{(x_3 - x_1)n_1 + x_3 - k}{x_2 - x_3} & \dfrac{(x_1 - x_2)n_1 + k - x_2}{x_2 - x_3} & d-1 \\ 1 & -1 & -(k - x_1)/(x_2 - x_3) & (k - x_1)/(x_2 - x_3) & dn_1 - v + 1. \end{array}$$

Proof. This follows from the preceding remarks and Theorem 1.1. $\square$

Note that if $p_{21}^3 = 1$, then Γ_2 is also not connected, so we can write $\mathscr{B} = \mathscr{B}_1' \cup \mathscr{B}_2' \cup \cdots \cup \mathscr{B}_{d'}'$, where any two blocks of $\mathscr{B}_i'$ intersect in x_2 points, and $\mathscr{B}_i$ intersects $\mathscr{B}_j'$ in one block. Any block is in a unique such intersection, so labeling the blocks by pairs (i, j), $(1 \leq i \leq d,\ 1 \leq j \leq d'$, we can see that this is the rectangular lattice scheme $R(d, d')$. However, by Proposition 5.2, that scheme is never the block configuration of a c.c. of type $(2, 2; 4)$, and therefore $p_{21}^3 > 1$.

There is a lot of structure inherent in these c.c.'s.

Lemma 6.5. *Let F_i be the incidence relation of the associated design restricted to $X_1 \times \mathscr{B}_i$. $(X_i, \mathscr{B}_i, F_i)$ is a dual 2-design if $x_1 \neq 0$.*

Proof. The incidence structure $(X_1, \mathscr{B}_i, F_i)$ clearly has v points, $n_1 + 1$ blocks, k points per block, and a point is in $N_1 + 1$ blocks of $\mathscr{B}_i$, therefore this is a 1-design. Any two blocks of $\mathscr{B}_i$ intersect in x_1 points, therefore, if $x_1 \neq 0$, it is a dual 2-design. □

If $x_1 = 0$, then $P_1 = 1$, and therefore intersection size 0 defines a parallelism on the blocks. Then the blocks of the 1-design defined above are disjoint, so in particular $n_1 + 1 = v/k$.

Lemma 6.6. *The incidence structure $(\mathscr{B}_i, \mathscr{B}_j, F_{ij})$, $i \neq j$, where F_{ij} is f_2 restricted to $\mathscr{B}_i \times \mathscr{B}_j$, is a 1-design.*

Proof. For given block $B \in \mathscr{B}_i$, there are p_{21}^3 blocks C of $\mathscr{B}_j$ with $(B, C) \in f_2$, and the same holds interchanging i and j, so the incidence structure is a 1-design. □

Proof of Theorem 6.3. We will show that the sets $X_1, \mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_d$ together with the relations F_i and F_{ij} form a system of linked symmetric designs.

Since $v = |\mathscr{B}_i|$, we have $x_1 \neq 0$ and $(X_1, \mathscr{B}_i, F_i)$ is a symmetric 2-design for all i. Then for $i \neq j$, $(X_1, \mathscr{B}_i, F_i)$ and $(X_1, \mathscr{B}_j, F_j)$ are symmetric designs on the same point set such that any block of $\mathscr{B}_i$ intersects any block of $\mathscr{B}_j$ in one of the two intersection sizes.

Clearly $(X_1, \mathscr{B}_i \cup \mathscr{B}_j, F_i \cup F_j)$ is a 2-design. We define relations on the blocks by the restriction of the relations on all the blocks. We will show that the conditions (2.1)–(2.4) hold, and thus the design defines a c.c. of type $(2, 2; 4)$.

Clearly (2.1) and (2.2) hold. If $p \in X_1$, and $B \in \mathscr{B}_i \cup \mathscr{B}_j$, then the number of blocks $C \in \mathscr{B}_i \cup \mathscr{B}_j$ containing p with $(B, C) \in f_1$ is N_1 or P_1 as $p \in B$ or not, and (2.3) holds. If B_1 and B_2 are blocks in $\mathscr{B}_i \cup \mathscr{B}_j$, then number of blocks C with $(B_1, C) \in f_1$ and $(B_2, C) \in f_1$ is $n_1 - 1$ if $(B_1, B_2) \in f_1$, and 0 otherwise. Therefore (2.4) holds, and the design is one associated with a c.c. of type $(2, 2; 4)$.

Now given $B_1, B_2 \in \mathscr{B}_i$, the number of $C \in \mathscr{B}_j$ with $(B_1, C) \in f_2$ and $(B_2, C) \in f_2$ is a constant, so the incidence structure $(\mathscr{B}_i, \mathscr{B}_j, F_{ij})$ is a 2-design which is obviously symmetric. The remaining conditions for a system of linked symmetric designs are easily checked. □

If the design associated with the c.c. of type $(2,2;4)$ is quasi-symmetric, so $x_1 = x_3$, then $p_{22}^3 = p_{22}^1 = p_{21}^3(p_{21}^3 - 1)(d-1)/n_1$. Since any two blocks in the same class intersect in x_1 points, the resolution defined by the imprimitivity classes is inner.

Proposition 6.7. *If $\mathscr{C}$ is a c.c. of type $(2, 2; 4)$ with $p_{11}^2 = p_{11}^3 = 0$ and $x_1 = x_3$, then the imprimitivity classes define an inner σ-resolution (which is not outer) of the associated design. Conversely, any inner σ-resolution (which is not outer) of a quasi-symmetric design defines a $(2, 2; 4)$ refinement with $p_{11}^2 = p_{11}^3 = 0$, by taking f_1 to be the resolution.*

Proof. We have already shown the first part of the proposition.

Suppose $X_2 = \mathscr{B}_1 \cup \mathscr{B}_2 \cup \cdots \cup \mathscr{B}_d$ is an inner σ-resolution of a quasi-symmetric design (X_1, X_2, F) with intersection sizes a and b, such that any two blocks in the same class intersect in b points. Define a refinement by

$$f_1 = \{(B, C)\colon B \text{ and } C \text{ are in the same class}\},$$
$$f_2 = \{(B, C)\colon |B \cap C| = a\},$$
$$f_3 = \{(B, C)\colon |B \cap C| = b,\ B \text{ and } C \text{ are in different classes}\}.$$

Then conditions (2.1)–(2.4) are easily checked, and this is a $(2, 2; 4)$ refinement of the quasi-symmetric design. Note that the resolution cannot be outer or we would have $f_3 = \emptyset$. □

Ray-Chauduri and Wilson in [16] investigated the existence of designs with σ-resolutions. They prove the following theorem.

Theorem 6.8 [16]. *Given $k \geqslant 2$, there exists a constant $C(k)$ such that if $v \geqslant C(k)$ and $v \equiv k \pmod{k(k-1)}$, then there exists a 2-$(v, k, 1)$ design with a 1-resolution.*

They also point out that $C(3) = C(4) = 1$. A 1-resolution is clearly inner, so many examples of inner resolutions of quasi-symmetric designs exist.

Case 3: $p_{21}^3 = p_{11}^3 = 0$, *three intersection sizes.*

The designs based on an affine design and a quasi-symmetric design (Example 6) are of this type.

Let $\mathscr{B}_1, \mathscr{B}_2, \ldots, \mathscr{B}_d$ be the imprimitivity classes. Let Γ_{1i} be the connected component of Γ_1 corresponding to $\mathscr{B}_i$. Our main result is the following.

Theorem 6.9. (1) *Γ_{1i} is a connected strongly regular graph with parameters $(n_1 + n_2 + 1, n_1, p_{11}^1, p_{11}^2)$ and has connected complement.*

(2) *Let F_i be the incidence f_6 restricted to $X_1 \times \mathscr{B}_i$. $(X_1, \mathscr{B}_i, F_i)$ is a 1-design with two block intersection sizes which is not a 2-design.*

This contrasts with the previous case, where the corresponding structure $(X_1, \mathscr{B}_i, F_i)$ is a 2-design in the only example we know of, the designs based on a system of linked symmetric designs.

Proof. From $p_{11}^2 \neq 0$, $p_{11}^3 = 0$, we have that two blocks intersect in x_1 or x_2 points if and only if they are in the same class. Hence $|\mathcal{B}_i| = n_1 + n_2 + 1$. Two blocks are adjacent in Γ_{1i} if they intersect in x_1 points, so the graph is regular with valency n_1. The number of vertices adjacent to two adjacent vertices is p_{11}^1, and the number adjacent to two non-adjacent vertices is p_{11}^2. Therefore the graph is strongly regular, and connected since $p_{11}^2 \neq 0$.

The complement of Γ_{1i} is strongly regular with parameters $(n_1 + n_2 + 1, n_2, p_{22}^2, p_{22}^1)$. Assume $p_{22}^1 = 0$, that is, the complement is not connected. $p_{22}^1 = 0$ and $p_{21}^1 = 0$ imply $p_{21}^2 = 0$, $p_{31}^2 = 0$, and $p_{11}^2 = n_1$. Then from (4.7), $x_1 N_1 + (k - x_2)P_1 = x_1 n_1$, and it follows that $k^2 = x_1 v$.

However, consider a point p and a block $B \in \mathcal{B}_i$. If $p \in B$, there are $N_1 + N_2 + 1$ blocks of $\mathcal{B}_i$ containing p; if $p \notin B$, there are $P_1 + P_2$ blocks of $\mathcal{B}_i$ containing p; and both types of blocks occur, so $1 + N_1 + N_2 = P_1 + P_2$. Then by (4.3) and (4.4), $P_3 = N_3$ which implies $k^2 = x_3 v$, and the design is quasi-symmetric, a contradiction; and we have shown (1).

$(X_1, \mathcal{B}_i, F_i)$ has k points per block and a given point is in $1 + N_1 + N_2$ blocks, so it is a 1-design with v points and $1 + n_1 + n_2$ blocks. Note that if p is a point, B a block, then there are N_i blocks containing p intersecting B in x_i points if $p \in B$ and P_i if $p \notin B$, for $i = 1, 2$.

The argument that $k^2 = x_3 v$ holds in general for case 3, which implies that $p_{22}^3 = 0$. We can therefore interchange f_1 and f_2 if necessary and assume $x_1 > x_2$.

Let G_{1i} be the adjacency matrix of Γ_{1i}, and suppose G_{1i} has eigenvalues n_1, r, and s with multiplicities 1, f and g respectively, where $r > 0$ and $s < 0$. Now we can write $A_1 = \operatorname{diag}(G_{11}, G_{12}, \ldots, G_{1d})$, hence A_1 has eigenvalues n_1, r and s with multiplicities d, df and dg respectively. From the character-multiplicity table for the block configuration, we have therefore that $\theta_{i1} = n_1$, and $z_i = d - 1$, for some $i \in \{1, 2, 3\}$.

Suppose $(X_1, \mathcal{B}_i, F_i)$ is a 2-design, so it is quasi-symmetric. Then the block graph for intersection size x_1 is Γ_{1i}, and it follows that $r = N_1 - P_1$ and $f = v - 1$, see [10]; so $\theta_{11} = r$ and $z_1 = df = d(v - 1)$. But $z_1 = v - 1$, so d must be 1, a contradiction. Therefore $(X_1, \mathcal{B}_i, F_i)$ cannot be a 2-design. □

We will finish by giving the rank 4 intersection matrices and character-multiplicity tables for these designs, insofar as they can be determined. We start with a lemma on intersection sizes.

Lemma 6.10. *Either $x_1 > x_3 > x_2$, or $x_2 > x_3 > x_1$.*

Proof. Assume $x_1 > x_3$ and $x_2 > x_3$. Then $x_1 v - k^2 > x_3 v - k^2 = 0$, so

$$N_1 - P_1 = n_1(x_1 v - k^2)/(k(v - k)) > 0,$$

and similarly $N_2 - P_2 > 0$. Thus $N_1 - P_1 + N_2 - P_2 \geq 2$; but $N_1 - P_1 + N_2 - P_2 =$

$P_3 - N_3 - 1 = -1$, contradiction. In the same way, if $x_1 < x_3$ and $x_2 < x_3$, then $N_1 - P_1 + N_2 - P_2 \leqslant -2$, again a contradiction. □

We can use this to determine the eigenvalues of A_1 in terms of the eigenvalues r and s of Γ_{1i}. Assume $x_1 > x_3 > x_2$.

If $\theta_{11} = n_1$, then

$$N_1 - P_1 = n_1(x_1 v - k^2)/(k(v-k)) = n_1,$$

so $k = x_1$, but we are assuming no repeated blocks. $\theta_{11} = N_1 - P_1 > 0$, so θ_{11} must be r and $z_1 = df$. We also know $z_1 = v - 1$, so d divides $v - 1$.

θ_{21} and θ_{31} are numbered so $\theta_{21} \geqslant \theta_{31}$, hence $\theta_{21} = n_1$, $\theta_{31} = s$, $z_2 = d - 1$, and $z_3 = dg$.

The intersection matrices of the block configuration can be expressed in terms of n_1, n_2, and p_{11}^2 as follows, where $n_3 = (d-1)(n_1 + n_2 + 1)$:

$$\begin{matrix} 0 & 1 & 0 & 0 \\ n_1 & n_1 - \dfrac{p_{11}^2 n_2}{n_1} & p_{11}^2 & 0 \\ 0 & \dfrac{p_{11}^2 n_2}{n_1} & n_1 - p_{11}^2 & 0 \\ 0 & 0 & 0 & n_1 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 1 & 0 \\ 0 & \dfrac{p_{11}^2 n_2}{n_1} & n_1 - p_{11}^2 & 0 \\ n_2 & n_2 - \dfrac{p_{11}^2 n_2}{n_1} & n_2 + p_{11}^2 - n_1 - 1 & 0 \\ 0 & 0 & 0 & n_2 \end{matrix}$$

$$\begin{matrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & n_1 \\ 0 & 0 & 0 & n_2 \\ n_3 & n_3 & n_3 & (d-2)(n_1 + n_2 + 1). \end{matrix}$$

The character-multiplicity table for the block configuration is

$$\begin{array}{cccc|c} 1 & n_1 & n_2 & n_3 & 1 \\ 1 & r & -(r+1) & 0 & v-1 \\ 1 & n_1 & n_2 & -n_1 - n_2 - 1 & d-1 \\ 1 & s & -(s+1) & 0 & b - v - d + 1. \end{array}$$

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